Solutions to slides of the 29^{th}

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First example

The first example is a separable equation :

$$\begin{aligned}
\dot{y} := \frac{dy}{dt} &= \frac{1}{t^2}(y+1) \\
\Leftrightarrow \frac{dy}{y+1} &= \frac{dt}{t^2} \\
\Leftrightarrow \int \frac{dy}{y+1} &= \int \frac{dt}{t^2} \\
\Leftrightarrow \ln(y+1) &= \frac{-1}{2t} + c \\
\Leftrightarrow e^{\ln(y+1)} &= e^{\frac{-1}{2t}+c} \\
\Leftrightarrow y+1 &= \underbrace{e^c}_{=:d} e^{\frac{-1}{2t}} \\
\Leftrightarrow y &= de^{\frac{-1}{2t}} - 1
\end{aligned}$$

This yields an infinite number of candidate solutions, each of them depending on the value of d. However, we know that $y(0) = 1$, from which we deduce :

$$1 = de^{\frac{-1}{2t}} - 1$$

which has no finite solution at $t = 0$ (since $e^{1/t} \to \infty$ as $t \to 0$). Hence, there is no solution.

Second example

This is a non-homogenous, linear system of equation.

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The first thing that should strike you is that the matrix of coefficients is *already* diagonal. Hence, there is no need to find the eigenvectors/values.

In fact, for any diagonal matrix, one should realise that eigenvalues are -1 times the elements of the diagonal and that eigenvectors are the unit vectors. So, in our case, it turns out that we have $\lambda_1 = 0.25, \lambda_2 = 0.5$ and $v_1 = [1, 0]', v_2 = [0, 1]'$. Since both eigenvalues are larger than zero, we can already deduce that the solution around the unique equilibrium candidate will be unstable.

That being said, we use the formula we derived in the first tutorial to find the solution :

$$\begin{aligned} \dot{y} &= a(t)y(t) + x(t) \\ \Rightarrow y &= \left[\int e^{-\int a(t)dt} x(t)dt + c \right] e^{\int a(t)dt} \end{aligned}$$

For our first equation, we have $a(t) = -0.25$ and $x(t) = t$. From this, we find :

$$\begin{aligned} e^{\int a(t)dt} &= e^{-0.25t} & e^{-\int a(t)dt} &= e^{0.25t} \\ \int e^{-\int a(t)dt} x(t)dt &= \int e^{0.25t} t dt & &\text{(integrate by parts)} \\ &= 4e^{0.25t}t - 4\int e^{0.25t}dt \\ &= 4e^{0.25t}t - 16e^{0.25t} + c \end{aligned}$$

Hence, the general solution to the upper equation of this problem is :

$$\begin{aligned} y_1 &= \left[\ 4e^{0.25t}t - 16e^{0.25t} + c\right] e^{-0.25t} \\ &= 4t - 16 + ce^{-0.25t} \end{aligned}$$

And since $y_1(0) = 1$, we find :

$$\begin{aligned} 1 &= -16 + c \\ \Leftrightarrow c &= 17 \end{aligned}$$

Hence, $y_1 = 4t - 16 + 17e^{-0.25t}$. As we can see, $\lim_{t \to \infty} y_1 = \infty$, as the eigenvalues of the system already told us.

For the second line $y_2 = -0.5y_2 + t^2$, we have $a(t) = -0.5$ and $x(t) = t^2$. Thus, we find :

$$e^{\int a(t)dt} = e^{-0.5t} \qquad\qquad e^{-\int a(t)dt} = e^{0.5t}$$

$$\begin{aligned}
\int e^{\int a(t)dt} x(t)dt &= \int e^{0.5t} t^2 dt && \text{(integrate by parts)} \\
&= 2e^{0.5t}t^2 - 4 \int e^{0.5t} t dt && \text{(integrate by parts again)} \\
&= 2e^{0.5t}t^2 - 8e^{0.5t}t + 8 \int e^{0.5t} dt \\
&= 2e^{0.5t}t^2 - 8e^{0.5t}t + 16e^{0.5t} + c
\end{aligned}$$

Hence, the general solution is :

$$\begin{aligned}
y_2 &= \left[\ 2e^{0.5t}t^2 - 8e^{0.5t}t + 16e^{0.5t} + c\right] e^{-0.5t} \\
&= 2t^2 - 8t + 16 + ce^{0.5t}
\end{aligned}$$

and since $y_2(0) = 1$, we have that :

$$\begin{aligned}
1 &= 16 + c \\
\Leftrightarrow c &= -15
\end{aligned}$$

Hence, we have the solution :

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 17 & 0 \\ 0 & -15 \end{pmatrix} \begin{pmatrix} e^{-0.25t} \\ e^{-0.5t} \end{pmatrix} + \begin{pmatrix} 0 & 4 & -16 \\ 2 & -8 & 16 \end{pmatrix} \begin{pmatrix} t^2 \\ t \\ 1 \end{pmatrix}$$

A quick check to see if our solution verifies the original problem :

$$\begin{aligned}
\dot{y}_1 &= 17 \times -0.25 \times e^{-0.25t} + 4 \\
&= -0.25 \left[17 \times e^{-0.25t} - 16\right] \\
&= -0.25 \left[17 \times e^{-0.25t} - 16 + 4t - 4t\right] \\
&= -0.25 \left[17 \times e^{-0.25t} - 16 + 4t\right] + t \\
&= -0.25 y_1 + t && \text{(yes)} \\
\dot{y}_2 &= -15 \times -0.5 \times e^{-0.5t} + 4t - 8 \\
&= -0.5 \left[\ -15e^{-0.5t} - 8t + 16 + 2t^2 - 2t^2\right] \\
&= -0.5 \left[\ -15e^{-0.5t} - 8t + 16 + 2t^2\right] + t^2 \\
&= -0.5 y_2 + t^2 && \text{(how surprising)}
\end{aligned}$$

If you look at figure 1, you can see the dynamics of (y_1, y_2) for the first 400 units of time. y_2 is the vertical axis and note the difference in magnitude with y_1.

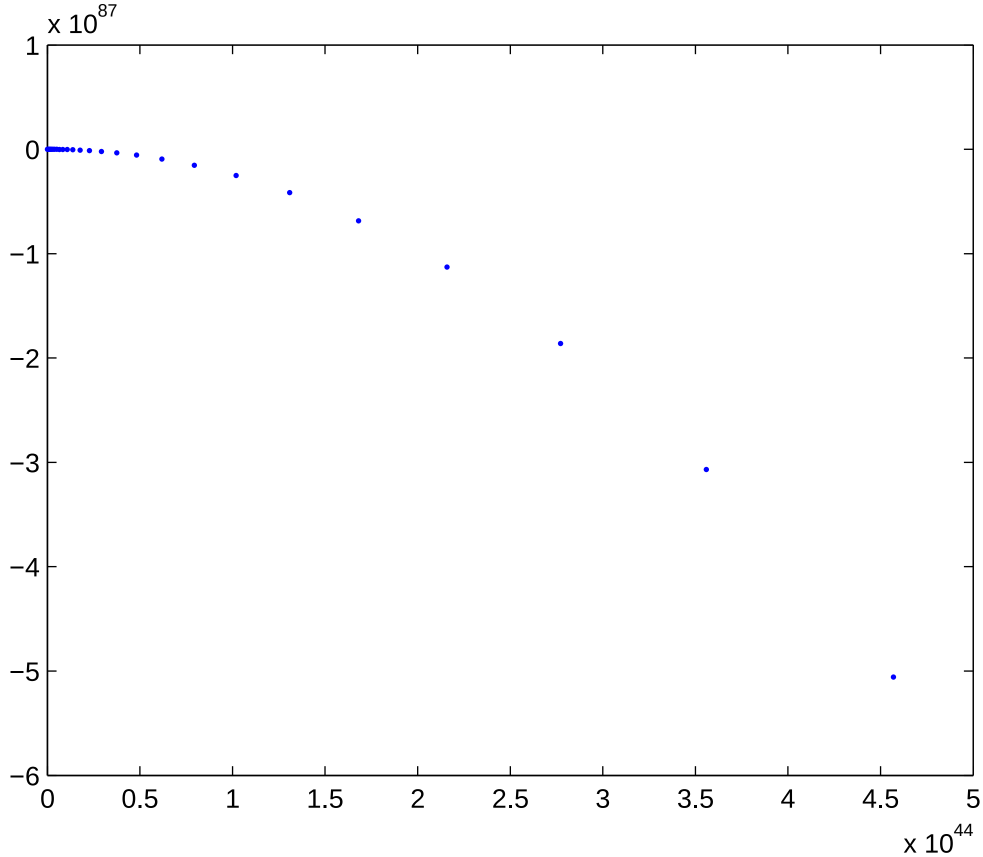

FIGURE 1 – **Solutions of the system of DE for $t = 1 \ldots 400$.**