

A solution to homework #2

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1)

a)

Note that $\hat{k} := \frac{K}{LH} = k$ where $L = e^{nt}$ and $H = e^{xt}$. With this assumption, we have :

$$\begin{aligned} \left(\frac{K}{LH}\right)^\alpha \left(\frac{LH}{LH}\right)^{1-\alpha} &= \underbrace{\frac{C}{LH}}_{=: \hat{c}} + \delta \frac{K}{LH} + \frac{\dot{K}}{LH} \\ \hat{k}^\alpha &= \hat{c} + \delta \hat{k} + \frac{\dot{K}}{LH} \end{aligned}$$

We need to find an expression for $\dot{\hat{k}}$, so that we can substitute $\frac{\dot{K}}{LH}$:

$$\begin{aligned} \dot{\hat{k}} &= \frac{\dot{K}}{LH} - \frac{\dot{L}}{L} \frac{K}{LH} - \frac{\dot{H}}{H} \frac{K}{LH} \\ &= \frac{\dot{K}}{LH} - (n+x)\hat{k} \\ \Rightarrow \dot{\hat{k}} + (n+x)\hat{k} &= \frac{\dot{K}}{LH} \end{aligned}$$

And thus, the resource constraint is :

$$\hat{k}^\alpha = \hat{c} + (\delta + n + x)\hat{k} + \dot{\hat{k}}$$

Which implies :

$$\hat{k}^\alpha - (\delta + n + x)\hat{k} - \hat{c} = \dot{\hat{k}}$$

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The growth of capital per effective labor (PEL) is equal to what is left of production once we deduce what is required to keep capital PEL constant (e.g. : $-(\delta + n + x)\hat{k}$) and what is consumed.

In a Solow world, consumption is exogenous and equal to $(1 - s)Y = (1 - s)K^\alpha L^{1-\alpha}$. From this, we deduce that consumption PEL is $(1 - s)\hat{y} = (1 - s)\hat{k}^\alpha$ and we then have :

$$s\hat{k}^\alpha - (\delta + n + x)\hat{k} = \dot{\hat{k}}$$

b)

Discard the trivial solution $\hat{k} = 0$ (meaningless in terms of economics). We must then have :

$$s\hat{k}^{\alpha-1} - (\delta + n + x) = \lambda_{\hat{k}}$$

Differentiation with respect to time yields :

$$s(\alpha - 1)\hat{k}^{\alpha-2}\dot{\hat{k}} = 0$$

Since $s \neq 0, \alpha \neq 1$ and that we discarded the case where $\hat{k} = 0$, we must have $\dot{\hat{k}} = 0$, which implies, that $\lambda_{\hat{k}} = 0$. This gives us a hint on a manner to have a non-zero growth rate in a steady state. Assume $\alpha = 1$ and then the above equation does not require $\dot{\hat{k}} = 0$ to hold. This actually yields a growth rate of :

$$s - (\delta + n + x) = \lambda_{\hat{k}}$$

which can have positive, negative (or zero) values. This is the case where the production function is linear in capital.

By definition, we have :

$$\begin{aligned} \lambda_{\hat{k}}\hat{k} &= \dot{\hat{k}} \\ \Rightarrow \lambda_{\hat{k}}\frac{K}{LH} &= \frac{\dot{K}}{LH} - (n + x)\frac{K}{LH} \end{aligned}$$

and since $\lambda_{\hat{k}} = 0$, we find :

$$\begin{aligned} (n + x)K &= \dot{K} \\ \Rightarrow (n + x) &= \lambda_K \end{aligned}$$

Hence capital grows at the combined rate of technology and population.

c)

k is defined as $\frac{K}{LH}$ and does not grow over time. Using the result found in b) :

$$\begin{aligned} s\hat{k}^{\alpha-1} - (\delta + n + x) &= 0 && \Leftrightarrow \\ \hat{k}^{\alpha-1} &= \frac{(\delta + n + x)}{s} && \Leftrightarrow \\ \hat{k}^* &= \left(\frac{\delta + n + x}{s} \right)^{\frac{1}{\alpha-1}} \end{aligned}$$

And since $k = \hat{k}$, we have the required result.

d)

Note that by the resource constraint, we have :

$$\begin{aligned} Y &= (1 - s)Y + \delta K + \dot{K} \\ \Rightarrow s\frac{Y}{K} &= \delta + \lambda_K \end{aligned}$$

Differentiation with respect to time and our result found on λ_K yields :

$$\begin{aligned} s\frac{\dot{Y}}{K} - s\frac{Y}{K}\frac{\dot{K}}{K} &= 0 \\ s\frac{\dot{Y}}{Y}\frac{Y}{K} - s\frac{Y}{K}\frac{\dot{K}}{K} &= 0 \end{aligned}$$

Since $s \neq 0, K \neq 0$, we then have that $Y \neq 0$ and thus, we can safely divide each side by $s\frac{Y}{K}$ to obtain :

$$\begin{aligned} \frac{\dot{Y}}{Y} &= \frac{\dot{K}}{K} = \lambda_K = (n + x) \\ &= n && \text{(since } x = 0 \text{ is assumed)} \end{aligned}$$

The growth rate of the economy is equal to the growth rate of capital. Hence the growth rate of $y = \frac{Y}{L}$ is :

$$\begin{aligned} \dot{y} &= \frac{\dot{Y}}{Y}\frac{Y}{L} - \frac{\dot{L}}{L}\frac{Y}{L} \\ \lambda_y &= (n + x) - n = 0 && \text{(since } x = 0 \text{ is assumed)} \end{aligned}$$

This is intuitive : capital/economy growth is solely driven by the population growth. Hence, GDP per person should be constant.

e)

We have :

$$s\hat{k}^\alpha - (\delta + n + x)\hat{k} = \dot{\hat{k}}$$

The system is non-linear, but we can use a change of variable to linearise it. Define $z = \frac{1}{\hat{k}^{\alpha-1}}$, we then have that $(1 - \alpha)\dot{z} = \frac{\dot{\hat{k}}}{\hat{k}^\alpha}$, which implies :

$$\frac{s}{1 - \alpha} - \frac{(\delta + n + x)}{1 - \alpha}z = \dot{z}$$

Solve the homogenous case first :

$$\begin{aligned}\frac{\dot{z}}{z} &= \frac{n + \delta + x}{\alpha - 1} \\ \Rightarrow \int \frac{dz}{z} &= \frac{n + \delta + x}{\alpha - 1} \int dt \\ \Rightarrow \ln z &= \frac{n + \delta + x}{\alpha - 1}t + c_0 \\ \Rightarrow z &= e^{\frac{n + \delta + x}{\alpha - 1}t} \underbrace{e_0^c}_{=: c_1} \\ &= c_1 e^{\frac{n + \delta + x}{\alpha - 1}t}\end{aligned}$$

We then solve for c_1 using :

$$\begin{aligned}
c_1 &= \frac{s}{1-\alpha} \int e^{\frac{n+x+\delta}{1-\alpha}t} dt \\
&= \frac{s}{n+x+\delta} e^{\frac{n+x+\delta}{1-\alpha}t} + c_2 \\
\Rightarrow z &= \left(\frac{s}{n+x+\delta} e^{\frac{n+x+\delta}{1-\alpha}t} + c_2 \right) e^{\frac{n+\delta+x}{\alpha-1}t} \\
&= \frac{s}{n+x+\delta} + c_2 e^{\frac{n+\delta+x}{\alpha-1}t} \\
&= \frac{s + c_3 e^{\frac{n+\delta+x}{\alpha-1}t}}{n+x+\delta} \quad (\text{where } c_3 := \frac{c_2}{n+x+\delta}) \\
\Rightarrow \frac{1}{\hat{k}^{\alpha-1}} &= \frac{s + c_3 e^{\frac{n+\delta+x}{\alpha-1}t}}{n+x+\delta} \\
\Rightarrow \hat{k}^{\alpha-1} &= \frac{n+x+\delta}{s + c_3 e^{\frac{n+\delta+x}{\alpha-1}t}} \\
\Rightarrow \hat{k}(t) &= \left(\frac{n+x+\delta}{s + c_3 e^{\frac{n+\delta+x}{\alpha-1}t}} \right)^{\frac{1}{\alpha-1}}
\end{aligned}$$

Finally, since we must have :

$$k_0 = \left(\frac{n+x+\delta}{s+c_3} \right)^{\frac{1}{\alpha-1}}$$

one can deduce that :

$$\frac{n+x+\delta}{k_0^{\alpha-1}} - s = c_3$$

Also note that since $\alpha < 1$, we have

$$\lim_{t \rightarrow \infty} \hat{k}(t) = \left(\frac{n+x+\delta}{s} \right)^{\frac{1}{\alpha-1}} = \hat{k}^*$$

You can see in figure 1 the solution drawn for $\delta = 0.05, x = 0, n = 0.01, \alpha = s = 0.3$ and $k_0 \in \{42, 0.06\}$ (values above and below the steady state). Convergence should be obvious.

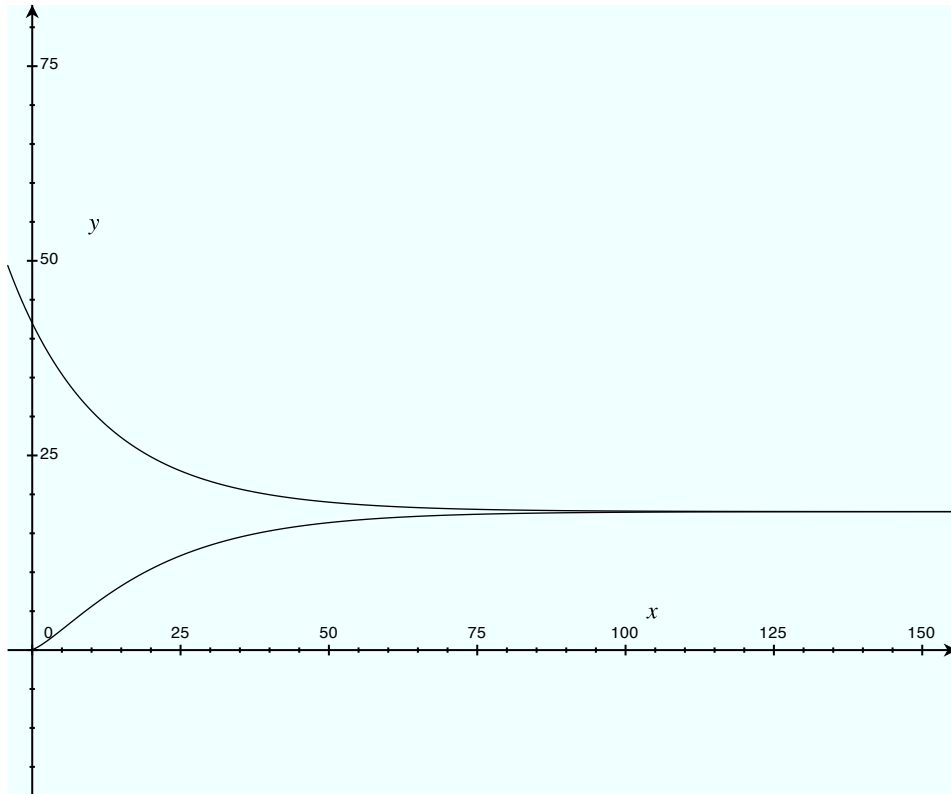


FIGURE 1 – $\hat{k}(t)$ as a function of time.

f)

The linear approximation of \dot{k} is :

$$\begin{aligned}\dot{k} &\approx \underbrace{\dot{k}^*}_{=0} + \left[s\alpha \hat{k}^{*\alpha-1} - (n+d+x) \right] (k - k^*) \\ &= 0 + \left[s\alpha \frac{n+x+\delta}{s} - (n+d+x) \right] (k - k^*) \\ &= 0 + (\alpha - 1)(n+x+\delta)(k - k^*) \\ \Rightarrow sp_k := \frac{\dot{k}}{k^* - k} &\approx (\alpha - 1)(n+x+\delta)\end{aligned}$$

From this, we can find an approximate value for $\frac{dsp_k}{ds}$:

$$\frac{dsp_k}{ds} \approx 0$$

g)

This is the Ramsey model. See Barro & Sala-i-Martin for a discussion on speed of convergence.

2)

1) and 3)

As always, we have :

$$\begin{aligned}\dot{K} &= Y - \delta K - C \\ \Rightarrow \frac{\dot{K}}{L} &= y - \delta k - c\end{aligned}$$

And

$$\begin{aligned}k &= \frac{K}{L} \\ \Rightarrow \dot{k} &= \frac{\dot{K}}{L} - \frac{\dot{L}}{L}k \\ \Rightarrow \dot{k} + n(t)k &= \frac{\dot{K}}{L}\end{aligned}$$

So

$$\begin{aligned}\dot{k} &= y - \delta k - c - n(t)k \\ &= sAk^\alpha - (\delta + n(t))k\end{aligned}$$

Since $n(t)$ varies with time, we find, substituting values for α, A, δ, s :

$$\dot{k} = \begin{cases} \frac{2}{10}k^{\frac{3}{4}} - 0.08k & \text{if } k < 5000 \\ \frac{2}{10}k^{\frac{3}{4}} - 400 & \text{if } 5000 \leq k < 40000 \\ \frac{2}{10}k^{\frac{3}{4}} - 0.01k & \text{if } 40000 \leq k \end{cases}$$

Intuition : rich (high capital) countries make less kids than poor countries.

Now, these values are terrible for a graph (and my graph is not to scale), but this should look like figure 2. This reveals multiple equilibria. Label them k_1^*, k_2^*, k_3^* . Note that if $k_1^* < k < k_2^*$, we have that $\dot{k} < 0$. Similarly, if $k_2^* < k < k_3^*$, we have $\dot{k} > 0$ and thus k^*2 is unstable. Now, if $0 < k < k_1^*$, $\dot{k} > 0$ and if $k > k_3^*$, $\dot{k} < 0$ and thus, k_1^* and k_2^* are stable equilibria.

Hence, if you start with low capital, you will stay with low capital (and vice-versa).

2)

If $y_0 \approx 53$, that means $k_0 \approx 200$ and you will end up in the low equilibrium since it is between k_1^* and k_2^* . Conversely, you will end up in the high equilibrium if $y_0 \approx 3080$, since $k_0 \approx 45000 \in [k_2^*, k_3^*]$. The growth rate for each of them will respectively be :

$$\begin{aligned}\gamma_1(k) &= 0.2k^{-1/4} - 0.08 \approx -26 \times 10^{-2} \\ \gamma_3(k) &= 0.2k^{-1/4} - 0.01 \approx -37 \times 10^{-3}\end{aligned}$$

3)

1)

Define $\hat{y} = \frac{Y}{L}$ as output PEL. We then have :

$$\begin{aligned}\hat{y} &= A \left(\frac{K}{L} \right)^\alpha \frac{H^\eta L^{1-\eta-\alpha}}{(L)^{1-\alpha}} \\ &= Ak^\alpha h^\eta\end{aligned}$$

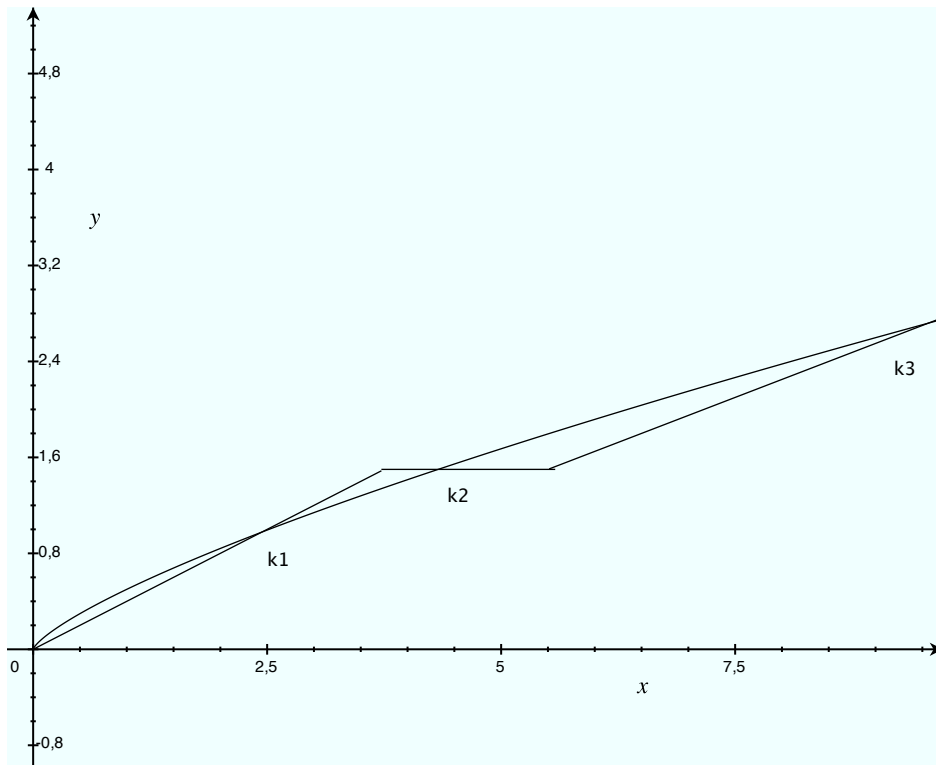


FIGURE 2 – Multiple equilibria (not to scale).

2)

Competitive markets implies that input are paid up to their marginal value of production. We have :

$$\begin{aligned}
 r_K + \delta_k &:= \frac{\partial Y}{\partial K} = \alpha \frac{Y}{K} \\
 &= \alpha \frac{y}{k} \\
 r_H + \delta_h &:= \frac{\partial Y}{\partial H} = \eta \frac{Y}{H} \\
 &= \eta \frac{y}{h} \\
 w &:= \frac{\partial Y}{\partial L} = (1 - \alpha - \eta)y
 \end{aligned}$$

Note that like any homogenous function of degree one, we have $Y = \frac{\partial Y}{\partial K}K + \frac{\partial Y}{\partial H}H + \frac{\partial Y}{\partial L}L$, which implies absence of pure profits¹.

3)

Note that $\ln Y = \alpha \ln K + \eta \ln H + (1 - \eta - \alpha) \ln L$ and thus :

$$\begin{aligned}
 \frac{\dot{Y}}{Y} &= \alpha \frac{\dot{K}}{K} + \eta \frac{\dot{H}}{H} + 0 \\
 \Rightarrow \frac{\dot{y}}{y} &= \alpha \frac{\dot{k}}{k} + \eta \frac{\dot{h}}{h} \quad (\text{since } \dot{L} = 0)
 \end{aligned}$$

Now, note that first order conditions for profit maximisation implies :

$$\begin{aligned}
 \alpha \frac{Y}{K} - \delta &= \eta \frac{Y}{H} - \delta && \Leftrightarrow \\
 \frac{\alpha}{\eta} H &= K && \Leftrightarrow \\
 \frac{\alpha}{\eta} h &= k && \Rightarrow \\
 \frac{\alpha}{\eta} \dot{h} &= \dot{k} && \Rightarrow \\
 \frac{\dot{k}}{k} &= \frac{\dot{h}}{h}
 \end{aligned}$$

1. This is another « Euler theorem ». If $y(x_1, \dots, x_n)$ is hd1, then $y = \sum_{i=1}^n \frac{\partial y}{\partial x_i} x_i$.

And thus, we can express the growth rate as :

$$\frac{\dot{y}}{y} = (\alpha + \eta) \frac{\dot{k}}{k}$$

Finally, note that the the growth rate of $k + h$ is given by the ressource constraint :

$$\begin{aligned} \dot{K} + \dot{H} &= sY - \delta(K + H) \\ \Rightarrow \dot{k} \left(1 + \frac{\alpha}{\eta}\right) &= sy - \delta k \left(1 + \frac{\alpha}{\eta}\right) \\ \Rightarrow \dot{k} &= s \frac{\eta}{\eta + \alpha} y - \delta k \\ \Rightarrow \frac{\dot{k}}{k} &= s \frac{\eta}{\eta + \alpha} \frac{y}{k} - \delta \\ &= s \frac{\eta}{\eta + \alpha} \left(\frac{\alpha}{\eta}\right)^\eta k^{\alpha-1+\eta} - \delta \end{aligned}$$

this allows us to do the usual phase diagram mumbo jumbo. Nothing really changes qualitatively.

4)

1)

$$\begin{aligned} r_r &:= -\frac{u''c}{u'} = \frac{\eta e^{-\eta c}}{e^{-\eta c}} = \eta c \\ \Rightarrow \epsilon &= -\frac{1}{\eta c} \end{aligned}$$

The bigger η , the more the function is concave (e.g. : the impact of marginal consumption becomes smaller). The more you consume, the less you are inclined to substitute present for future (see 3). This is a form of impatience by consumption (e.g. addiction).

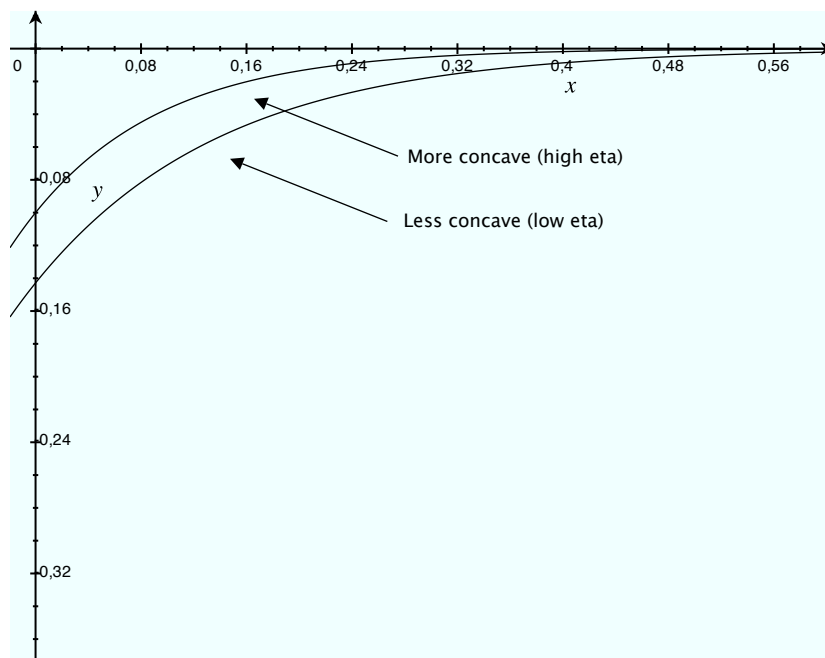


FIGURE 3 – Utility function for different values of η

2) and 3)

The central planner problem is (without explicit mention of time) :

$$\begin{aligned} & \max_C \int_0^\infty \left[-\frac{1}{\eta} e^{-\eta C} e^{n-\rho t} + \lambda \left(AK^\alpha e^{(1-\alpha)nt} - \delta K - C \right) \right] dt \\ \text{s.t. } & \dot{K} \leq AK^\alpha e^{(1-\alpha)nt} - \delta K - C \\ & C \geq 0 \\ & \lim_{t \rightarrow \infty} e^{-\bar{r}t} K \geq 0 \end{aligned}$$

Normalising the budget constraint for individuals

Utility is strictly increasing in c . Hence, the budget constraint will be binding ($\lambda \neq 0$).

$$\begin{aligned} \frac{\dot{K}}{L} &= AK^\alpha e^{-\alpha nt} - \delta \frac{K}{L} - \frac{C}{L} \\ &= Ak^\alpha - \delta k - c \end{aligned}$$

Now, we need to find \dot{k} . Since $k = K/L$, we have :

$$\begin{aligned} \dot{k} &= \frac{\dot{K}}{L} - \frac{\dot{L}}{L} \frac{K}{L} = \frac{\dot{K}}{L} - nk \\ \Rightarrow \dot{k} + nk &= \frac{\dot{K}}{L} \end{aligned}$$

So that :

$$\dot{k} = Ak^\alpha - (\delta + n)k - c$$

The rate of growth of capital per unit of labor is production minus the consumption and the amount of capital per person we must spend to keep the ratio of capital per person constant (e.g. to counteract depreciation and growth).

Perfect competition implies that :

$$\begin{aligned} Y &= wL + rK \\ w &= (1 - \alpha)Ak^\alpha \\ r &= \alpha Ak^{\alpha-1} - \delta \\ \Rightarrow y &= wh + rk \end{aligned}$$

where h are the hours worked by individuals. Since there is no disutility in working, we must conclude that $h^* = \max h$, which we define to be one. Hence $y = w + rk$. Now,

individuals maximise with respect to the assets they hold (a). Aggregation and absence of other assets implies that :

$$\begin{aligned} La &= K \\ \Rightarrow a &= k \end{aligned}$$

Hence, the individual resource constraint is :

$$\dot{a} = w + ra - na - c$$

All this implies the following OCP :

$$\begin{aligned} V &:= \max_c \int_0^\infty \frac{-1}{\eta} e^{-\eta c + (n-\rho)t} dt \\ \text{s.t. } c &\geq 0 \\ \dot{a} &= w + ra - na - c \\ 0 &\leq \lim_{t \rightarrow \infty} e^{-\int_0^\infty r(t)dt} a \end{aligned}$$

This implies the following Hamiltonian :

$$H := \frac{-1}{\eta} e^{-\eta c + (n-\rho)t} + \lambda [w + ra - na - c]$$

with c as a control variable and k as a state variable. Hence :

$$\begin{aligned} \frac{\partial H}{\partial c} = 0 &\Leftrightarrow e^{-\eta c} e^{(n-\rho)t} = \lambda \\ \frac{\partial H}{\partial a} = -\dot{\lambda} &\Leftrightarrow \lambda(r - n) = -\dot{\lambda} \end{aligned}$$

Note that the first equation implies :

$$\begin{aligned} (n - \rho)\lambda - \eta\dot{\lambda} &= \dot{\lambda} \\ \Rightarrow \rho + \eta\dot{c} &= r \end{aligned}$$

Hence, this leads to the two following differential equations :

$$\begin{aligned} \dot{c} &= \frac{1}{\eta}(r - \rho) && \text{(consumption growth)} \\ \dot{a} &= w + ra - na - c && \text{(assets growth)} \end{aligned}$$

Aggregation over all individuals allows us to substitute back a and r in terms of k :

$$\begin{aligned} \dot{c} &= \frac{1}{\eta}(\alpha Ak^{\alpha-1} - \delta - \rho) && \text{(consumption growth)} \\ \dot{k} &= Ak^\alpha - (n + \delta)k - c && \text{(assets growth)} \end{aligned}$$

Taken together, these two equations defines an optimal policy function (stable arm) for which the consumer will choose the optimal level of consumption leading to a unique steady state. The slope of this stable arm will be given by :

$$\frac{dc}{dk} = \frac{Ak^\alpha - (n + \delta)k - c}{\frac{1}{\eta}(\alpha Ak^{\alpha-1} - \delta - \rho)}$$

4)

If there is no growth, we have :

$$\begin{aligned} 0 &= \frac{1}{\eta}(\alpha Ak^{\alpha-1} - \delta - \rho) \\ 0 &= Ak^\alpha - (n + \delta)k - c \\ \Rightarrow k^* &= \left(\frac{\delta + \rho}{\alpha A} \right)^{\frac{1}{\alpha-1}} \\ \Rightarrow c &= Ak^\alpha - (n + \delta)k \end{aligned}$$

Now, note that if $c > Ak^\alpha - (n + \delta)k$, we have $\dot{k} < 0$ and conversely, $c < Ak^\alpha - (n + \delta)k \Rightarrow \dot{k} > 0$. Similarly, $k > k^* \Rightarrow \dot{c} < 0$ and $k < k^* \Rightarrow \dot{c} > 0$.

These lines are drawn in figure 4 with $\delta = 0.05, x = 0, n = 0.01, \rho = 0.05, \alpha = 0.3$ and $A = 1$. From this, we can already deduce saddle path stability.

Nonetheless, we linearise the system :

$$\begin{aligned} \begin{pmatrix} \dot{c} \\ \dot{k} \end{pmatrix} &\approx \begin{pmatrix} c^* \\ k^* \end{pmatrix} + \begin{pmatrix} 0 & -\frac{1}{\eta}A\alpha(1-\alpha)k^{*\alpha-2} \\ -1 & \alpha Ak^{*\alpha-1} - (n + \delta) \end{pmatrix} \begin{pmatrix} c - c^* \\ k - k^* \end{pmatrix} \\ &= \begin{pmatrix} c^* \\ k^* \end{pmatrix} + \begin{pmatrix} 0 & -\frac{1}{\eta}A\alpha(1-\alpha)\underbrace{\left(\frac{\delta + \rho}{A\alpha}\right)^{\frac{\alpha-2}{\alpha-1}}}_{=:e} \\ -1 & \rho - n \end{pmatrix} \begin{pmatrix} c - c^* \\ k - k^* \end{pmatrix} \end{aligned}$$

This leads to the following characteristic polynomial :

$$\begin{aligned} 0 &= -\frac{1}{\eta}e - (\rho - n)\lambda + \lambda^2 \\ \Rightarrow \lambda_i &= \frac{(n - \rho) \pm \sqrt{(\rho - n)^2 - 4\frac{1}{\eta}e}}{2} \end{aligned}$$

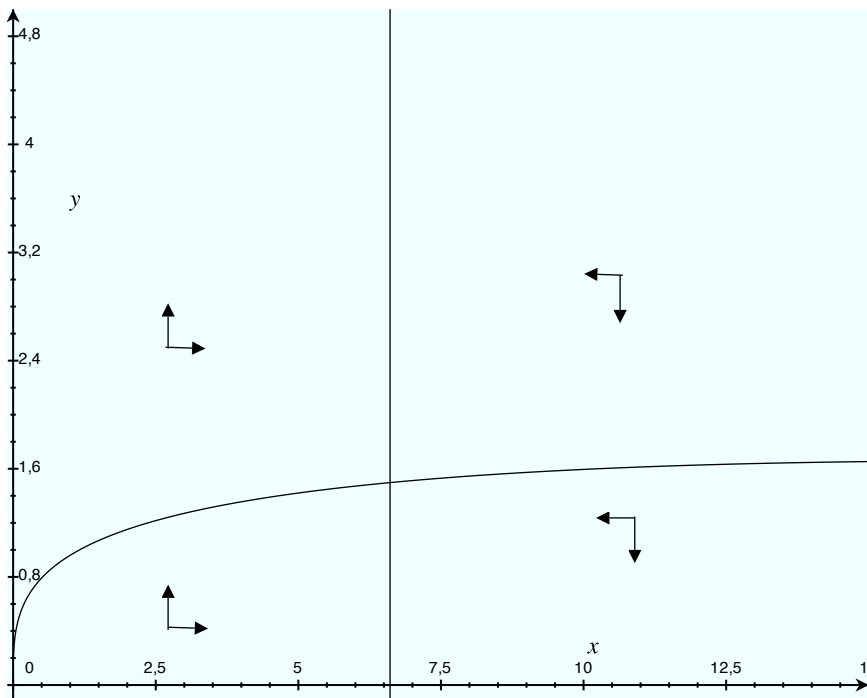


FIGURE 4 – This looks familiar...

Clearly, one value is negative ($n < \rho$). The other one is positive as we can deduce from :

$$\begin{aligned}\lambda_+ &= \frac{(n - \rho) + \sqrt{(\rho - n)^2 - 4\frac{1}{\eta}e}}{2} \\ &> \frac{(n - \rho) + \sqrt{(\rho - n)^2}}{2} \\ &= 0\end{aligned}$$

Hence, the system is saddle-path stable.

5)

Relabel the linearised system :

$$\begin{aligned}\dot{s} &= s^* + A(s - s^*) \\ &= (I - A)s^* + As\end{aligned}$$

with eigenvalues/vectors $\lambda_+/v_+, \lambda_-/v_-$ (+/- designates the positive / negative eigenvalue). From this, we define $Vs =: z$ and find :

$$\dot{z} = (I - D)z^* + Dz$$

wich can be solved to find $k(t)$ around the steady state.

We know that the solution to the homogenous system will be given by :

$$s = \begin{pmatrix} v_{+1} & v_{-1} \\ v_{+2} & v_{-2} \end{pmatrix} \begin{pmatrix} d_+ e^{\lambda_+ t} \\ d_- e^{\lambda_- t} \end{pmatrix}$$

for some constants d_+, d_- that depends on k_0 and the transversality condition. We also know that optimal control implies that we will be on the stable arm. Hence, around (c^*, k^*) , the change in Δc can be expressed by $\Delta k \times \frac{\Delta_- c}{\Delta_- k}$ (where $\frac{\Delta_- c}{\Delta_- k}$ is the slope of the negative eigenvector).

This will give us, for λ_- :

$$\frac{(n - \rho) + \sqrt{(\rho - n)^2 - 4\frac{1}{\eta}e}}{2} \Delta_- c - \frac{1}{\eta} e \Delta_- k = 0$$

leading to :

$$\begin{aligned}\frac{\Delta_- c}{\Delta_- k} &= \frac{2}{\eta} \frac{e}{(\rho - n) + \sqrt{(\rho - n)^2 - 4\frac{1}{\eta}e}} \\ &= \underbrace{\frac{e}{\eta}}_+ \underbrace{\frac{-1}{\lambda_-}}_+\end{aligned}$$

(note that this is consistent with our graph)

Thus, around the steady state, we can find the behavior of \dot{k} by :

$$\begin{aligned}\dot{k} &\approx k^* - (c - c^*) + (\rho - n)(k - k^*) \\ &\approx k^* - \frac{\Delta_- c}{\Delta_- k}(k - k^*) + (\rho - n)(k - k^*) \\ \Rightarrow \dot{k} &\approx k^* - \left[\frac{2}{\eta(\rho - n) + \sqrt{(\rho - n)^2 - 4\frac{1}{\eta}e}} + (\rho - n) \right] (k - k^*)\end{aligned}$$

This implies the following speed of convergence² :

$$sp_k := \frac{\dot{k}}{k^* - k} \approx -\frac{k^*}{k^* - k} + \left[\frac{2}{\eta(\rho - n) + \sqrt{(\rho - n)^2 - 4\frac{1}{\eta}e}} + (\rho - n) \right]$$

Luckily, k^* does not vary with η , so we find :

$$\begin{aligned}\frac{\partial sp_k}{\partial \eta} &\approx -\frac{e}{\eta^2} \frac{1}{\lambda_-^2} \frac{\partial \lambda_-}{\partial \eta} \\ &= -\underbrace{\frac{e}{\eta^2} \frac{1}{\lambda_-^2}}_+ \underbrace{\frac{1}{2\eta^2} \frac{4e}{\sqrt{(\rho - n)^2 - 4/\eta e}}}_+\end{aligned}$$

Thus, the speed of convergence varies negatively with η . This is very intuitive. The higher η , the less you are willing to substitute actual consumption for savings and thus, the less capital will build-up to reach the steady-state.

2. I use the same definition as in question one.