Tutorial 1 and 2 : Systems of differential equations

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Your TA...

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- probably makes a lot of grammatical errors and does not mind being corrected;
- has a very long name, so you can call him pabsta (my initials).

Derivatives with respect to time.

Definition (Basic notation and order)

Time derivatives have their own notation :

$$\dot{y} := \underbrace{\frac{dy}{dt}}_{\text{1st order}} \qquad \ddot{y} := \underbrace{\frac{d^2y}{dt^2}}_{\text{2nd order}}$$

and so on...

Differential equation

Definition (DE, Order, ODE, System)

- A differential equation (DE) is an equation with derivatives in it.
- Its order is the highest order of the derivatives in it.
- If the derivative is with respect to only one variable, then it is an ordinary differential equation (ODE).
- Many DE alltogether are called a system of DEs.

Examples

$$\ddot{a} + \dot{a} + \frac{\partial a}{\partial k} = 0 \qquad \text{(DE of 2nd order)}$$

$$\ddot{f} + \dot{f} + x = 0 \qquad \text{(ODE of 2nd order)}$$

$$\nabla U = \lambda p \qquad \text{(System of DE of 1st order)}$$

Differential equation.

Definition (Linear DE)

An (O)DE is linear if it is a linear combination of the derivatives.

Definition (Homogenous ODE)

An ODE is homogenous if it involves solely the derivatives of a variable and the variable itself[1].

[1] Note that this is not the definition of homogeneity as it is defined in microeconomics, for instance.

Examples

$$\ddot{a} + \dot{a} + \dot{a}^3 = 0 \qquad \text{(Non-linear DE)}$$

$$f(t)\ddot{a} + g(t)\dot{a} + h(t)\dot{a} = 0 \qquad \text{(Linear ODE)}$$

$$\ddot{f} + \dot{f} + f + x = 0 \qquad \text{(Non-homogenous)}$$

$$c_1\ddot{f} + c_2\dot{f} + c_3 f = 0 \qquad \text{(Homogenous)}$$

A more interesting example

From the Ramsey model (see
`http://www.pabsta.qc.ca/files/ps2.pdf`) :

$$\dot{k} = f(k) - c - (n+\delta)k \qquad \text{(Capital growth)}$$

$$\frac{\dot{c}}{c} = \frac{1}{\theta}(\underbrace{\frac{df(k)}{dk}}_{=:f'(k)} - \delta - \rho) \qquad \text{(Consumption growth)}$$

where n, δ, θ, ρ are parameters, $c := c(t), k := k(t)$ are respectively consumption per capital and capital per capita at time t. $f(k)$ is a per-capita production function.

A more interesting example

$$\dot{k} = f(k) - c - (n+\delta)k \qquad \text{(Capital growth)}$$

$$\frac{\dot{c}}{c} = \frac{1}{\theta}(f'(k) - \delta - \rho) \qquad \text{(Consumption growth)}$$

Except for particular cases on $f(k)$ that we do not really use, the first and second equations are non-linear.

Ways to solve (systems of) ODEs

- Analytically (powerful, but limited to some DE)
- Numerically (powerful, but stability is technique dependent)
- Graphically (good for intuition)

Analytically

We will look at :

- Linear, homogenous, first-order system of ODEs with constant coefficients.
- Linear, first-order system of ODEs with constant coefficients.
- Linear, first-order ODE with variable coefficients.

Linear, first-order system of ODEs with constant coefficients

Let the system be :

$$\begin{aligned} \dot{y}_1 - a_{11}y_1 - a_{12}y_2 - \cdots - a_{1n}y_n - x_1 &= 0 \\ \dot{y}_2 - a_{21}y_1 - a_{22}y_2 - \cdots - a_{2n}y_n - x_2 &= 0 \\ \vdots &= 0 \\ \dot{y}_n - a_{n1}y_1 - a_{n2}y_2 - \cdots - a_{nn}y_n - x_n &= 0 \end{aligned}$$

The key idea is to see this as a linear function from (and to) the space of differentiable functions. We can then think of vectors $\dot{\mathbf{y}} := [\dot{\mathbf{y}}_1, \dot{\mathbf{y}}_2, \ldots, \dot{\mathbf{y}}_n]', \mathbf{y}$ and $\mathbf{x}$ as vectors from this space.

Linear, first-order system of ODEs with constant coefficients

Thus the system is :

$$\dot{\mathbf{y}} = A\mathbf{y} + \mathbf{x}$$

the homogenous case is when $\mathbf{x} = \mathbf{0}$.

Linear, homogenous first-order system of ODEs with constant coefficients

The system is :

$$\dot{\mathbf{y}} = A\mathbf{y}$$

How do we solve this ? If the dimension of the system is one (e.g. $n = 1$), it is pretty obvious.

$n = 1$

$$\begin{aligned} && \frac{dy}{dt} &= a_{11}y \\ \Rightarrow && \frac{1}{y}dy &= a_{11}dt \\ \Leftrightarrow && \int \frac{1}{y}dy &= \int a_{11}dt \\ \Leftrightarrow && \ln y &= a_{11}t + c \\ \Leftrightarrow && y &= e^c e^{a_{11}t} \\ \Leftrightarrow && y &= de^{a_{11}t} \end{aligned}$$

where d has to be given by some *initial conditions*.

$n \geq 1$

Now what if the system is of greater size ? If the matrix A is *diagonal*, then, it is pretty much the same procedure as above applied n times.
It turns out that there is a very useful theorem of linear algebra that we can use to be in this case. To understand this, we need a little bit of vocabulary.

Eigenvectors, eigenvalues

Definition (Eigenvectors, Eigenvalues)

Let A be an invertible matrix. Then, we call $\mathbf{x}, \lambda$ respectively the eigenvectors and eigenvalues of A if they are a non trivial solution to the system :

$$A\mathbf{x} = \lambda\mathbf{x}$$

(or $(A - \lambda I)\mathbf{x} = \mathbf{0}$).

Eigenvectors, eigenvalues

- Vectors that satisfies this property keeps the same orientation (up to a minus sign) in space before or after the linear transformation. They are only streched by a factor of λ.
- There are n eigenvalues which are not necessarely different.
- Eigenvalues are different from zero and can be real or complex.
- There are n eigenvectors and they form a basis (they span the full space).

So, why do we care ?

A useful theorem

Theorem

Let A be an invertible matrix, then

$$V^{-1}AV = D$$

where D is the diagonal matrix of eigenvalues :

$$D := \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ 0 & \vdots & \vdots & 0 \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$$

and V is the matrix of eigenvectors :

$$V := [\mathbf{x_1}, \mathbf{x_2}, \dots, \mathbf{x_n}]$$

Still, why do we care ?

Solving for $n \geq 1$

Recall the system

$$\dot{\mathbf{y}} = \mathbf{A}\mathbf{y}$$

and define

$$\mathbf{z} := \mathbf{V}^{-1}\mathbf{y}$$

We then have :

$$\begin{aligned} & \dot{\mathbf{z}} = V^{-1}\dot{\mathbf{y}} \\ \Rightarrow \quad & V\dot{\mathbf{z}} = \dot{\mathbf{y}} \\ \Rightarrow \quad & \dot{\mathbf{z}} = V^{-1}AV\mathbf{z} \\ \Rightarrow \quad & \dot{\mathbf{z}} = D\mathbf{z} \end{aligned}$$

So we can solve for $\mathbf{z}$ by integrating n times and we can find back $\mathbf{y}$ by the identity $V\mathbf{z} = \mathbf{y}$. Using some conditions (transversality condition, initial conditions, steady state conditions, etc..), the system is then fully solved.

How do we find D and V ?

- Use your favorite paper technique to solve $A\mathbf{x} = \lambda\mathbf{x}$.
- Use a software to find a numerical approximation (in matlab use the function $[V, D] = eig(A)$)

Example

Let the system be described by

$$\dot{\mathbf{y}} = \underbrace{\begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}}_{=A} \mathbf{y}$$

$$\Rightarrow \qquad A - \lambda I = \begin{pmatrix} 1-\lambda & 0.5 \\ 0.5 & 1-\lambda \end{pmatrix}$$

For $(A - \lambda I)\mathbf{x} = \mathbf{0}$ for $x \neq \mathbf{0}$, we must have $\det(A - \lambda I) = 0$, which implies that $(1-\lambda)^2 = 0.25$ and yield the solutions $\lambda_1 = 1.5, \lambda_2 = 0.5$.

Example (continued)

Now, we find the eigenvectors by solving :

$$\begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

This leads to $\mathbf{v_1} = [\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}]'$ and $\mathbf{v_2} = [-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}]'$

Example (continued)

Hence the transformed system is :

$$\begin{aligned} \dot{\mathbf{z}} &= \underbrace{\begin{pmatrix} 1.5 & 0 \\ 0 & 0.5 \end{pmatrix}}_{=D} \mathbf{z} \\ \Rightarrow \qquad \ln \mathbf{z} &= \begin{pmatrix} 1.5t \\ 0.5t \end{pmatrix} + \mathbf{c} \\ \Rightarrow \qquad \mathbf{z} &= \begin{pmatrix} d_1 e^{1.5t} \\ d_2 e^{0.5t} \end{pmatrix} \end{aligned}$$

And since $\mathbf{y} = \mathbf{V}\mathbf{z}$, we have

$$\mathbf{y} = \begin{pmatrix} \frac{\sqrt{2}}{2} d_1 e^{1.5t} - \frac{\sqrt{2}}{2} d_2 e^{0.5t} \\ \frac{\sqrt{2}}{2} d_1 e^{1.5t} + \frac{\sqrt{2}}{2} d_2 e^{0.5t} \end{pmatrix}$$

Linear, first-order system of ODEs with constant coefficients

Now, what if $\mathbf{x} \neq \mathbf{0}$?

$$\dot{\mathbf{y}} = A\mathbf{y} + \mathbf{x}$$

Well, we use the same trick and define $\mathbf{z} = \mathbf{V}^{-1}\mathbf{y}$ so that the system becomes :

$$\dot{\mathbf{z}} = D\mathbf{z} + \mathbf{V}^{-1}\mathbf{x}$$

which can be solved piecewise and we can then find back $\mathbf{y} = \mathbf{V}\mathbf{z}$ (we will see how later).

Eigenvalues and economic interpretation

In economics, we look for *stable* equilibriums. What is the link between equilibriums an eigenvalues ? We can see that they appear as time multipliers in the exponents of solutions. For the matter of the discussion, assume $d_1 = d_2 = 1$.

Eigenvalues and economic interpretation

Then, for real numbers :

- If the eigenvalues are greater than zero, the system will explode to infinity.
- If the eigenvalues are lower than zero, the system will converge to a stable solution which depends on $V^{-1}\mathbf{x}$.

Of course, if d_1, d_2 have some different values (perhaps zero), things might change according to their value, but the reasonning is similar.

Eigenvalues and economic interpretation

Now suppose the system is instead :

$$\dot{\mathbf{y}} = \begin{pmatrix} 1 & 0.5 \\ -0.5 & 1 \end{pmatrix} \mathbf{y}$$

This leads, in order to find the eigenvalues, to the following system to solve $(1 - \lambda)^2 = -0.25$. which has no real solutions. Yet, strangely, the system *still* has a solution. How do we find it ?

An heuristical introduction to complex numbers

- If we work with the naturals and try to solve $x + 2 = 1$, there is no solution. So we introduce negative numbers (the integers).
- If we work with the integers and try so solve $3x + 1 = 2$, there is no solution. So we introduce rational numbers (fractions).
- If we work with the rationals and try so solve $x^2 = 2$, there is no solution. So we introduce irrational numbers (to form the real line).
- If we work with the real line and try so solve $x^2 = -1$, there is no solution. So we introduce ...

Complex numbers

Definition (imaginary unit)

We define i, the imaginary unit, as the solution to $i^2 = -1$.

Definition (Complex number)

Let a, b be any real numbers. Then, $(a + bi)$ is a complex number.

Algebra on complex numbers

Let $(a + bi)$, $(c + di)$ be two complex numbers. Then :

- $(a + bi) + (c + di) = (a + b) + (c + d)i$ (substraction is defined likewise).
- $(a+bi)(c+di) = ab+adi+cbi+bd(i^2) = (ab-bd)+(ad-cb)i$
  .
- $\frac{(a+bi)}{(c+di)} = \frac{(a+bi)}{(c+di)}\frac{(c-di)}{(c-di)} = \frac{1}{c^2+d^2}(a + bi)(c - di)$

Two important theorems

We will use, but not prove the following :

Theorem (Fundamental theorem of algebra)

Let $y = \sum_{j=0}^{n} a_j x^j$ be a polynomial of degree n with a_j real numbers. Then there exists, n complex roots which solves $y = 0$.

Theorem (Euler's theorem)

Let r be a real number, then $e^{ir} = \cos(r) + i\sin(r)$.

In Laymen's terms

Theorem (Fundamental theorem of algebra)

Polynomials always have solutions in the set of complex numbers. Hence, eigenvalues always exists in the set of complex numbers.

Theorem (Euler's theorem)

If the imaginary part moves in time, it can be thought as rotating the real part around a circle.

Eigenvalues and economic interpretation

Hence, $(1-\lambda)^2 = -0.25$ has two complex roots, which are $\lambda_1 = 1 + 0.5i, \lambda_2 = 1 - 0.5i$ so we can solve the system and we will find a solution that looks like :

$$\mathbf{y} = \begin{pmatrix} a_{11} e^t e^{0.5ti} + a_{12} e^t e^{-0.5ti} \\ a_{21} e^t e^{0.5ti} + a_{22} e^t e^{-0.5ti} \end{pmatrix}$$

And using Euler :

$$\mathbf{y} = \begin{pmatrix} a_{11} e^t \left[\cos(0.5t) + i\sin(0.5t)\right] + a_{12} e^t \left[\cos(-0.5t) + i\sin(-0.5t)\right] \\ a_{21} e^t \left[\cos(0.5t) + i\sin(0.5t)\right] + a_{22} e^t \left[\cos(-0.5t) + i\sin(-0.5t)\right] \end{pmatrix}$$

Eigenvalues and economic interpretation

Since we only observe the real part, we forget the imaginary one and we find :

$$\mathbf{y}^* = \begin{pmatrix} a_{11}e^t\cos(0.5t) + a_{12}e^t\cos(-0.5t) \\ a_{21}e^t\cos(0.5t) + a_{22}e^t\cos(-0.5t) \end{pmatrix}$$

Eigenvalues and economic interpretation

For the sake of the discussion, let $\lambda_1 = (f + bi)$, $\lambda_2 = (c + di)$ be the eigenvalues. Then, the observed system is :

$$\mathbf{y}^* = \begin{pmatrix} a_{11}e^{ft}\cos(bt) + a_{12}e^{ct}\cos(dt) \\ a_{21}e^{ct}\cos(dt) + a_{22}e^{ft}\cos(bt) \end{pmatrix}$$

Interpretation

- If the real component of complex numbers is smaller than one, the system converges to an equilibrium. Otherwise, it may diverge to $\pm$ infinity or be zero, depending on the values of the a's.
- The imaginary part acts induces a circular motion of the solution. It can spin clockwise or counterclockwise, depending on the sign. The magnitude of the imaginary parts determines the frequency of this rotation.
- Thus, we would then observe a divergent or convergent spiral, depending on the signs and magnitudes.
- What to remember ? Complex solutions can still lead to convergent (stable) equilibriums and introduce *cyclicality*.

Solving linear ODEs with variable coefficients

- Here we focus on a single equation that looks like

$$\dot{y} = a(t)y(t) + x(t)$$

- For example, a resources constraint can have the form $\dot{k} = w(t)k + r(t)k - \delta k - c(t)$ where $c(t)$ can be thought as a known function.
- This can be solved by finding the solution to the homogenous case first (e.g. : with $x(t) = 0$) and then solve the general case.

In details

We solve first the homogenous case :

$$\begin{aligned} \frac{dy}{dt} &= a(t)y(t) \\ \Rightarrow \qquad \frac{dy}{y} &= a(t)dt \\ \Rightarrow \qquad y(t) &= de^{\int a(t)dt} \end{aligned}$$

Then, the general solution must be of that form since otherwise, the homogenous would have no be solution. Hence, the only difference is that d must be a function of time when $x(t) \neq 0$.

In details (continued)

Hence, we must have :

$$\begin{aligned} & a(t)y(t) + x(t) = \dot{d}e^{\int a(t)dt} + a(t)de^{\int a(t)dt} \\ \Rightarrow \quad & x(t) = \dot{d}e^{\int a(t)dt} \\ \Rightarrow \quad & e^{-\int a(t)dt}x(t) = \dot{d} \\ \Rightarrow \quad & \int e^{-\int a(t)dt}x(t)dt + c = d \end{aligned}$$

where c is some integration constant.
Thus, the general solution is :

$$\left[\int e^{-\int a(t)dt}x(t)dt + c \right] e^{\int a(t)dt}$$

An (easy) example

Let $\dot{y} = \frac{1}{t}y + 32t$. Here, $x(t) = 32t$ and $a(t) = 1/t$. All we have to do is to find the different parts of the previous formula :

$$\begin{aligned} & \int a(t)dt = \ln(t) \\ \Rightarrow \quad & e^{\int a(t)dt} = t \\ \Rightarrow \quad & e^{-\int a(t)dt} = 1/t \\ \Rightarrow \quad & \int e^{-\int a(t)dt} x(t)dt = 32t \\ \Rightarrow \quad & y(t) = (32t + c)t = 32t^2 + ct \end{aligned}$$

Simple differentiation allows us to verify the solution.

Solving non-linear ODEs

So far we worked with linear ODEs. Most ODEs in economics are non-linear. Unfortunately, unless we deal with some special cases, there are no known analytical methods to solve those systems. What do we do ?

- We linearize the system (projections on a specific interval, Taylor expansions, etc.)
- This is done around some interesting value, generally a steady state, which leads to a system of the form $\dot{y} = A(y - y^*)$ where $y^*$is the steady state.
- We can also use softwares for numerical solutions.
- We can also perform graphical analysis.

Using software

- Softwares like Matlab have numerical routines that find solutions (ODE23 and ODE45).
- Solutions found can be quite unstable and might depend on how accurate inital conditions are specified or the range of equations. Stability depends on how "sharp" is the system around the solution.

The time elimination method

Let's get back to our consumption/capital example with $f(k) = k^\alpha$
:

$$\dot{k} = \frac{dk}{dt} = k^\alpha - c - (n+\delta)k$$
$$\dot{c} = \frac{dc}{dt} = \frac{1}{\theta}(\alpha k^{\alpha-1} - \delta - \rho)c$$

If we use the implicit function theorem and divide the second equation by the first, we find :

$$\frac{dc}{dt}\frac{dt}{dk} = \frac{dc}{dk} = \theta \frac{k^\alpha - c - (n+\delta)k}{(\alpha k^{\alpha-1} - \delta - \rho)c}$$

The time elimination method

$$\frac{dc}{dt}\frac{dt}{dk} = \frac{dc}{dk} = \theta\frac{k^{\alpha} - c - (n+\delta)k}{(\alpha k^{\alpha-1} - \delta - \rho)c}$$

- By doing this, we got rid of time derivatives. The result is a non-linear differential equation that can be solved to find a policy function $c(k)$.
- There is only one possible problem (in this case), that is if we have a form $0/0$ in the equation above in the initial conditions.
- It turns out this is actually the case and we must resolve this indeterminancy by some trick (l'Hospital's rules, for instance).

Graphical analysis

Let's use the two of equations of the previous example again.

$$\dot{k} = \frac{dk}{dt} = k^{\alpha} - c - (n+\delta)k$$
$$\dot{c} = \frac{dc}{dt} = \frac{1}{\theta}(\alpha k^{\alpha-1} - \delta - \rho)c$$

- When $\dot{k} = 0$, the capital does not change anymore through time. Hence, the equation

$$k^{\alpha} = c + (n+\delta)k$$

  determines a set of points for which capital does not grow over time for any amout of consumption.
- Similarly, when $\dot{c} = 0$, we find

$$\alpha k^{\alpha-1} = \delta + \rho$$

  which determines a level of capital for which consumption does not grow.
- Note that this can be done for any system of DE.

Graphical analysis

If we graph these equations for given parameters, we find :

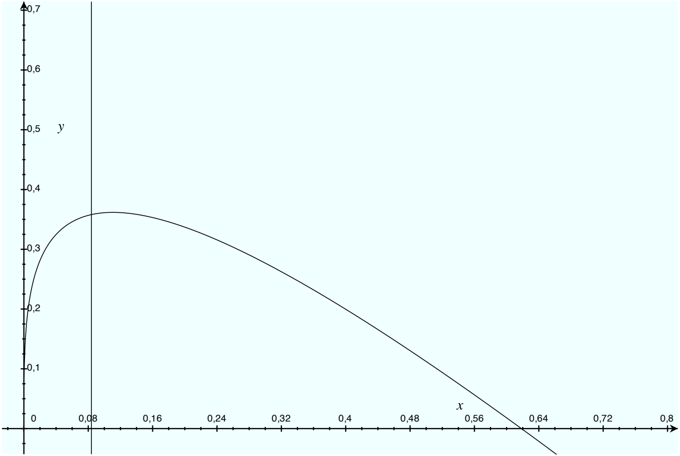

Figure: **Stable solutions to the system of DE.**

Graphical analysis

- It seems very natural to define the steady state where the two curves meet.
- The problem is that there are other candidates.
- The origin satisfies the two equations.
- Moreover, the point where $c = 0$ in the first equation is also a solution.

Which one do we pick ?

Graphical analysis

- To rule out some bad candidates, we need to refer to the conditions specified by the problem (initial conditions and the transversality condition). These two eliminates bad candidates.
- Intuitively, this makes sense. In the equilibrium at the origin, your consumption is zero, so everybody dies. In the second equilibrium where $k = 0$, nobody saves and there is no production in the next "instantaneous period" and everybody dies because they have nothing to consume.

Graphical analysis

Now, what is the dynamics of the system ?

- If $c < k^{\alpha} - (n+\delta)k$, capital grows over time and the system $(c(t), k(t))$ goes west over time. Graphically, this happens when we are below the loci $c = k^{\alpha} - (n+\delta)k$. If we are above, it is the opposite.
- If $k > \left(\frac{\delta+\rho}{\alpha}\right)^{1/(\alpha-1)}$, consumption grows over time and the system $(c(t), k(t))$ goes north over time. Graphically, this happens when we are on the left of the equation $k = \left(\frac{\delta+\rho}{\alpha}\right)^{1/(\alpha-1)}$. On the other side, it is the opposite.

Graphical analysis

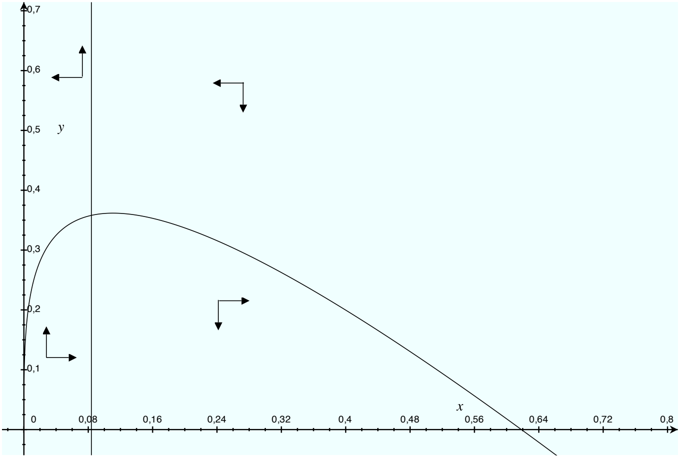

Figure: **Dynamics to the system of DE.**

Graphical analysis

How do we find the saddle path ?

Saddle path

- If we look back at the graphic, it should be intuitive that there exists a curve $c(k)$ between the two curves for which the system of DE will lead to the steady state. That is, the amount of consumption that balances perfectly for a correct capital/consumption growth, given an actual level of capital.
- This is called the saddle-path. It is also called the policy function.
- It is implicitely defined by the equation found by the time elimination method and the equilibrium values. Thus, it depends on the functionnal form of the system.